On some models of many agent systems with competition and cooperation

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Abstract

We review and develop a selection of models of systems with competition and cooperation, with origins in economics, where deep insights can be obtained by the mathematical methods of game theory. Some of these models were touched upon in authors' book 'Understanding Game Theory', World Scientific 2010, where also the necessary background on games can be found.

Key words: Bi-matrix inspection games, territorial price building, tax evasion, von Neumann - Morgenstern set, two-action multi-agent games, nonlinear Markov games, colored (or rainbow) options, geometry of risk-neutral laws

We review and develop below some models of competition and cooperation, with origins in economics. These models are linked only ideologically and methodologically, which allows one to read all sections almost independently.

1 Territorial price building

Here we present a variation of the classical Cournot model of price building, where selling and production sites are spatially distributed.

Suppose there is a market that include selling sites $M_1, \ldots, M_m$, products $1, \ldots, K$ and production sites $1, \ldots, L$.

There are N agents that buy the products at the production sites and sell them in $M_1, \ldots, M_m$.

Assume that at the each production site l there is a large stock of all products available at the price p_{kl} for the unit of the product k, and that the transportation cost to M_i of a unit of the product k is ξ_{ikl}.

Let Y_{ikl}^n denote the amount of the product k, which the agent n brought to M_i from the site l. Then the total amount of kth product in M_i is

$$Y_{ik} = \sum_{n=1}^{N} Y_{ik}^n, \quad Y_{ik}^n = \sum_{l=1}^{L} Y_{ikl}^n.$$

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The selling price of the product k in M_i is the function of the total supply of the product. Following the standard Cournot model assume that this price is given by the formula

$$R_{ik}(Y_{ik}) = (1 - Y_{ik}/\alpha_{ik})\beta_{ik}, \quad Y_{ik} \in [0, \alpha_{ik}],$$

with some positive constants α_{ik}, β_{ik}.

Then the total income of the player n can be calculated by the formula

$$H_n = \sum_{i=1}^{m} \sum_{k=1}^{K} H_n(ik) \tag{1}$$

with

$$H_n(ik) = Y_{ik}^n \left(1 - \frac{Y_{ik}}{\alpha_{ik}}\right)\beta_{ik} - \sum_{l=1}^{L} Y_{ikl}^n(\xi_{ikl} + p_{kl}), \quad Y_{ik} \in [0, \alpha_{ik}]. \tag{2}$$

Thus we have defined a symmetric N-person game Γ^N with the payoffs given by (1) and with the strategies of each player n being the arrays (Y_{ikl}^n) with positive entries bounded by α_{ik}.

For simplicity we reduce our attention to the case of 2 players.

Proposition 1.1. *The game Γ^2 has a symmetric Nash equilibrium. If for each pair (i, k) there exists a unique $l = q(i, k)$ where the minimum of $\xi_{ikl} + p_{kl}$ is attained, then this equilibrium is unique and the corresponding strategies of each player are given by*

$$\hat{Y}_{ik} = \frac{1}{3}\delta_{q(i,k)}^l\alpha_{ik}\left(1 - \frac{\xi_{ikl} + p_{kl}}{\beta_{ik}}\right), \tag{3}$$

where δ_m^l denotes the usual Dirac symbol that equals 1 for $l = m$ and zero otherwise.

Proof. Clearly looking for a best reply one has to maximize each $H_n(ik)$ separately for each pair (ik). It is also clear from (2) that to have a best reply one has to bring the product only from those sites that minimize $\xi_{ikl} + p_{kl}$. Hence given (Y_{ik}^1) to find the best reply of the second player one has to find the maximum of the functions

$$\bar{H}_2^{ik}(Y_{ikq(i,k)}^2) = Y_{ikq(i,k)}^n\left(1 - \frac{Y_{ik}^1 + Y_{ikq(i,k)}^2}{\alpha_{ik}}\right)\beta_{ik} - Y_{ikq(i,k)}^n(\xi_{ikq(i,k)} + p_{kq(i,k)})$$

for each pair (ik). Differentiating one gets

$$\frac{d\bar{H}_2^{ik}(z)}{dz} = \left(1 - \frac{Y_{ik}^1 + z}{\alpha_{ik}}\right)\beta_{ik} - (\xi_{ikq(i,k)} + p_{kq(i,k)}) - z\frac{\beta_{ik}}{\alpha_{ik}} = 0$$

for the best reply $z = Y_{ikq(i,k)}^2$, yielding

$$Y_{ikq(i,k)}^2 = \frac{1}{2}\alpha_{ik}\left(1 - \frac{Y_{ik}^1}{\alpha_{ik}} - \frac{\xi_{ikq(i,k)} + p_{kq(i,k)}}{\beta_{ik}}\right). \tag{4}$$

As the second derivatives of $\bar{H}_2^{ik}$ is negative this point does really define a maximum (and not a minimum). By the symmetry one concludes that symmetric equilibrium strategy solves the equation

$$\hat{Y}_{ik} = \frac{1}{2}\alpha_{ik}\left(1 - \frac{\hat{Y}_{ik}}{\alpha_{ik}} - \frac{\xi_{ikq(i,k)} + p_{kq(i,k)}}{\beta_{ik}}\right) \tag{5}$$

and thus equals (3) as required. □

2 Bi-matrix inspection games

In this and the next sections we develop some extensions of the classical inspection game. For other similar models see [1], [2], [5], [10] and references therein.

The game is carried out between a trespasser (player I) and an inspector(player II) in n-periods (or steps). We shall analyze a 'stop to abuses' version of this game when it lasts only till the first time trespassing is discovered by the inspector.

Player I has 2 pure strategies: to break (violate) the law (B) or to refrain from it (R). Player II has also 2 pure strategies: to check the actions of player I (C) or to have a rest (R). If player I chooses (R) he gets the legal income $r > 0$. If he chooses (B), he obtains additionally the illegal surplus $s > 0$. However, if his illegal action is discovered by player II, the player I pays the fine $f > 0$ and the game is over.

If player II chooses (C) he spends the amount $c > 0$ on this procedure and can discover the trespassing of player I with the probability $p(\overline{p} = 1 - p)$. If player I breaks the law and this action is not discovered, the inspector loses the amount $l > 0$.

Consequently one step of this game can be described by the table

		Inspector	
		Check (C)	Rest (R)
Trespasser	Break (B)	$-pf + \overline{p}(r+s), -(c+\overline{p}l)$	$r+s, -l$
	Refrain (R)	$r, -c$	$r, 0$

or shortly by the matrix

$$\begin{pmatrix} -pf + \overline{p}(r+s), -(c+\overline{p}l) & r+s, -l \\ r, -c & r, 0 \end{pmatrix} \tag{6}$$

It is natural to assume, that $c < pl$, so that the pair (B,R) is not a Nash equilibrium (otherwise the inspector has no reasons at all to conduct checks).

Let us say that a bi-matrix game *has a value*, if payoffs in any of the existing Nash equilibrium are the same, the corresponding pair of payoffs being called the *value of the game*.

Proposition 2.1. *Assume $f, r, s, c, l > 0$, $c < pl$. Then the game with the matrix (6) has a value $V = (u, v)$. Moreover, 1) if $s < s_1 = \frac{p}{\overline{p}}(f + r)$, the unique Nash equilibrium is given by the pair of mixed strategies $(x, \overline{x})$, $(y, \overline{y})$, where*

$$x = \frac{c}{pl}, y = \frac{s}{p(f+r+s)}; \quad u = r, v = -\frac{c}{p}.$$

2) if $s > s_1$, the unique Nash equilibrium is the profile (B,C) and

$$u = -pf + \overline{p}(r+s), v = -(c+\overline{p}l);$$

3) if $s = s_1$, then $u = r, v = -(c+\overline{p}l)$ and the Nash equilibria are given by all pairs (X, C), where X is any (pure or mixed) strategy of the trespasser.

Proof. Clearly the only candidate for Nash equilibrium in pure strategies is the pair (B, C). This profile is an equilibrium if and only if $s \geq s_1$. Formulae for x, y follow from the standard expression for mixed strategy profiles in two-action two-player games. Other statements are checked by a straightforward inspection. □

Of course of greater interest is the analysis of a multi-step version of this game. Let us consider the n-step game, where during this time the player I can break the law at most k times and the player II can organize the check at most m times. Assume that after the end of each period (step), the result becomes known to both players. Total payoff in n steps equals the sum of payoffs in each step. It is also assumed that all this information (rules of the game) is available to both players.

Let us denote the game described above by $\Gamma_{k,m}(n)$. Let $(u_{k,m}(n), v_{k,m}(n))$ be the value of this game. We then get the following system of recurrent equations:

$$(u_{k,m}(n), v_{k,m}(n))$$

$$= Val \begin{pmatrix} -pf + \overline{p}(r + s + u_{k-1,m-1}(n-1)), & r + s + u_{k-1,m}(n-1), \\ -(c + \overline{p}l) + \overline{p}v_{k-1,m-1}(n-1) & -l + v_{k-1,m}(n-1) \\ r + u_{k,m-1}(n-1), -c + v_{k,m-1}(n-1) & r + u_{k,m}(n-1), v_{k,m}(n-1) \end{pmatrix} \quad (7)$$

(if all $\Gamma_{k,m}(n)$ have values, i.e. their equilibrium payoffs are uniquely defined), with the boundary conditions $(m, n, k \geq 0)$:

$$(u_{0,m}(n), v_{0,m}(n)) = (nr, 0); \quad (8)$$

$$(u_{k,0}(n), v_{k,0}(n)) = (nr + ks, -kl); \quad k \leq n, \quad (9)$$

reflecting the following considerations: if the trespasser is unable to break the law, the pair of solutions (R,R) will be repeated over all periods; and if the inspector is unable to check, the trespasser will commit the maximum number of violations available.

Though $k \leq n, m \geq n$, the form of the recurrent equations below is slightly simplified if one allows all non-negative k, m, n together with the agreement

$$(u_{k,m}(n), v_{k,m}(n)) = (u_{k',m'}(n), v_{k',m'}(n)); k' = \min(k, n), m' = \min(m, n) \quad (10)$$

Let us reduce our attention further to the game $\Gamma_{n,n}(n)$. Let us write $U_n = u_{n,n}(n)$ and $V_n = v_{n,n}(n)$. Then (7) takes the form:

$$(U_n, V_n) = (U_{n-1}, V_{n-1}) + Val M_n, \quad (11)$$

where

$$M_n = \begin{pmatrix} \overline{p}(r + s) - p(f + U_{n-1}), -(c + \overline{p}l + pV_{n-1}) & r + s, -l \\ r, -c & r, 0 \end{pmatrix} \quad (12)$$

$$(n \geq 0; \quad U_0 = V_0 = 0)$$

For $n = 1$ the game becomes the same as the game (6), and its solution is given by Proposition 2.1.

Let us find the solution to the game $\Gamma_{2,2}(2)$. Plugging the values U_1 and V_1 into M_2 yields

$$M_2 = \begin{pmatrix} \overline{p}(r + s) - p(f + r), -\overline{p}l & r + s, -l \\ r, -c & r, 0 \end{pmatrix}, \quad 0 < s < s_1;$$

$$M_2 = \begin{pmatrix} \overline{p}(-pf + \overline{p}(r + s)), -\overline{p}(c + \overline{p}l) & r + s, -l \\ r, -c & r, 0 \end{pmatrix}, \quad s > s_1;$$

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$$s_2 = \frac{p}{\overline{p}}(f + r) + \frac{p}{\overline{p}^2}r.$$

Direct calculations show that under the assumptions of Proposition 2.1 the M_2 also has a value, and one can distinguish three basic cases (equilibrium strategies are again denoted by $(x, \overline{x})$, $(y, \overline{y})$):

1) if $0 < s < s_1$, then

$$x = \frac{c}{pl + c}, y = \frac{s}{p(f + 2r + s)}; \quad U_2 = 2r, V_2 = -\frac{c(2pl + c)}{p(pl + c)}$$

2) if $s_1 < s < s_2$, then

$$x = \frac{c}{p(c + (1 + \overline{p})l)}, \quad y = \frac{s}{p(\overline{p}f + (1 + \overline{p})(r + s))};$$

$$U_2 = -pf + \overline{p}(r + s), \quad V_2 = -\left(\frac{cl}{p(c + (1 + \overline{p})l)} + c + \overline{p}l\right)$$

3) if $s > s_2$, then

$$U_2 = (1 + \overline{p})(-pf + \overline{p}(r + s)), \quad V_2 = -(1 + \overline{p})(c + \overline{p}l),$$

and the profile (B,C) is an equilibrium.

Analogously one can calculate the solutions for other $n > 2$.

3 Tax payer against a tax man

This is a game between a tax payer (player I) and the tax police (player II). Player I has 2 pure strategies: to hide part of the taxes (H) or to pay them in full (P). Player II has also 2 strategies: to check player I (C) and to rest (R). Player I gets the income r if he pays the tax in full. If he chooses the action (H), he gets the additional surplus l. But if he is caught by player II, he has to pay the fine f.

In the profile (C,H) player II can discover the unlawful action of player I with the probability $p(\overline{p} = 1 - p)$, so that p can be called the efficiency of the police. Choosing (C), player II spends c on the checking procedure. Of course $l, r, f, c > 0$.

Hence we defined a bi-matrix game given by the table

		Player II (Police)	
		Check (C)	Rest (R)
Player I	Hide (H)	$r + \overline{p}l - pf, -c + pf - \overline{p}l$	$r + l, -l$
	Pay (P)	$r, -c$	$r, 0$

or shortly by the payoff matrix

$$\begin{pmatrix} r + \overline{p}l - pf, -c + pf - \overline{p}l & r + l, -l \\ r, -c & r, 0 \end{pmatrix} \tag{13}$$

The candidates to the mixed equilibrium are the strategies $(\beta, \overline{\beta})$, $(\alpha, \overline{\alpha})$, where

$$\alpha = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{l}{p(l + f)} > 0$$

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$$\beta = \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}} = \frac{c}{p(l+f)} > 0.$$

In order to have these strategies well defined, it is necessary to have $\alpha < 1$ and $\beta < 1$ respectively. By a direct inspection one gets the following result.

Proposition 3.1. *1) If $c \geq p(f + l)$, the pair (H,R) is an equilibrium, and moreover the strategy (R) is dominant for the police (even strictly, if the previous inequality is strict). 2) If $c < p(f + l)$ and $fp \leq \overline{p}l$, the pair (H,C) is an equilibrium and the strategy (H) is dominant (strictly if the previous inequality is strict). 3) If $c < p(f + l)$, $fp > \overline{p}l$, then the unique Nash equilibrium is the profile of mixed strategies $(\beta, \overline{\beta}), (\alpha, \overline{\alpha})$.*

Remarks. 1. Consequently, in cases 1) and 2) the actions of the police are not effective. 2. It is not difficult to show that the equilibrium in case 3) is stable.

It is more interesting to analyze the game obtained by extending the strategy space of player I by allowing him to choose the amount l of tax evasion: $l \in [0, l_M]$, where l_M is the full tax due to player I. For example, we shall assume that the fine is proportional to l, i.e. $f(l) = nl$. Say, in the Russian tax legislation $n = 0.4$. Under these assumptions the key coefficients α, β take the form

$$\alpha = \frac{1}{p(n+1)}, \quad \beta = \frac{c}{l} \frac{1}{p(n+1)}.$$

Let $H_I(l)$ denote the payoff to player I in the equilibrium when l is chosen. One can distinguish two cases:

1) $p > \frac{1}{n+1} \Longleftrightarrow \alpha < 1$. If

$$l > l_1 = \frac{c}{p(n+1)} \Longleftrightarrow \beta < 1,$$

then $(\beta, \overline{\beta}), (\alpha, \overline{\alpha})$ is a stable equilibrium. If $l < l_1 \Longleftrightarrow \beta > 1$, then (H, R) is a stable equilibrium. Since

$$H_I(l < l_1) = r + l < r + l_1,$$

$$H_I(l > l_1) = \beta\alpha(r + \overline{p}l - pf) + \beta\overline{\alpha}(r + l) + \overline{\beta}\alpha r + \overline{\beta}\overline{\alpha}r = r + \beta l(\alpha\overline{p} + \overline{\alpha} - \alpha pn),$$

it follows that $H_I(l > l_1) > H_I(l < l_1)$ would be possible whenever $\beta l(\alpha\overline{p} + \overline{\alpha} - \alpha pn) \geq l_1$. However $\alpha\overline{p} + \overline{\alpha} - \alpha pn = 0$, hence this is not the case. Consequently $H_I(l > l_1) < H_I(l < l_1)$ and therefore player I will avoid tax on the amount $l = l_1$.

2) $p < \frac{1}{n+1} \Longleftrightarrow \alpha > 1$.

If $l > l_1 \Longleftrightarrow \beta < 1$, then (H, C) is an equilibrium, and if $l < l_1 \Leftrightarrow \beta > 1$, then (H, R) is an equilibrium. Since

$$H_I(l < l_1) = r + l < r + l_1, \quad H_I(l > l_1) = r + (1 - p)l - pln,$$

the choice $l > l_1$ is reasonable for player I as

$$l(1 - p - pn) \geq l_1.$$

Hence $H_I(l > l_1) > H_I(l < l_1)$ whenever $l \geq \frac{l_1}{1 - p(n+1)}$.

Consequently, if

$$\frac{l_1}{1 - p(n+1)} \leq l_M, \tag{14}$$

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the equilibrium strategy for player I is $l = l_M$ and otherwise $l = l_1$.

One can conclude that in both cases it is profitable to avoid tax on the amount l_1, but as the efficiency of tax man increases, it becomes unreasonable to avoid tax on a higher amount.

Let us see which condition in the second case would ensure the inequality (14) when the amount of tax avoidance is l_M in the equilibrium. Plugging l_1 in (14) yields

$$\frac{c}{p(n+1)(1 - p(n+1))} \leq l_M.$$

Denoting $x = p(n+1) < 1$ one can rewrite it as

$$x^2 - x + \frac{c}{l_M} \leq 0. \tag{15}$$

The roots of the corresponding equation are

$$x_{1,2} = \frac{1 \pm \sqrt{1 - \frac{4c}{l_M}}}{2}.$$

Hence for $c > l_M/4$ inequality (14) does not hold for any p, and for $c \leq l_M/4$ the solution to (15) is

$$x \in \left[\frac{1 - \sqrt{1 - \frac{4c}{l_M}}}{2} ; \frac{1 - \sqrt{1 + \frac{4c}{l_M}}}{2} \right].$$

Thus for

$$c \leq \frac{l_M}{4}, \quad p \in \left[\frac{1 - \sqrt{1 - \frac{4c}{l_M}}}{2(n+1)} ; \frac{1 - \sqrt{1 + \frac{4c}{l_M}}}{2(n+1)} \right] \tag{16}$$

it is profitable to avoid tax payment on the amount l_M.

Let us consider a numeric example with $n = 0.4$ so that $1/(n+1) = 0.714$. If, say, $c = 1000$, then $l_M = 100000$.

1) If $p < 0.714$, the condition $c \leq l_M/4$ holds true, as $1000 < 2500$. Hence, for $p \in [0.007; 0.707]$ it is profitable to avoid tax on the whole amount, i.e. 100000.

2) If $p > 0.714$ it is profitable to avoid tax on the amount $l_1 = 714.29$.

Hence if the efficiency of tax payment checks is $p < 0.707$, it is profitable to avoid tax on the whole amount of 100000, and if $p > 0.707$, then not more than on 1010.

4 Cooperative games versus zero-sum games

The aim of this Section is to present a curious connection between competition and cooperation by linking the solutions to cooperative games and lower values of auxiliary zero-sum games. The following definitions are standard:

Cooperative game with non-transferable utility is a triple $G = (I, v, H)$, where $I = \{1, 2, \ldots, n\}$ is the set of players, H is a non-empty compact set from R^n, and v is a mapping from the set of all coalitions (non-empty subset $S \subset I$) to the set of non-empty closed subsets $v(S) \subset H$.

For $x, y \in H$ one says that x *dominates* y $(x \succ y)$, if there exists a coalition $S \subseteq I$ such that

1) $x, y \in v(S)$;
2) $x_i > y_i$ for any $i \in S$.

A *(von Neumann - Morgenstern) solution* to $G = (I, v, H)$ is called a subset $V \subseteq H$ such that

1) *(internal stability)* there are no pairs of vectors from V such that one of them dominates the other one;

2) *(external stability)* for any $y \in H \backslash V$ there exists $x \in V$ such that $x \succ y$.

For $A \subset R^n$ let us denote by A_ε the ε-neighborhood of A, i.e. $A_\varepsilon = A + B_\varepsilon$ with

$$B_\varepsilon = \left\{ x \in R^n \,\middle|\, \sum_{i=1}^{n} x_i^2 < \varepsilon \right\}.$$

A closed subset V is called a ε-*solution* whenever V is internally stable and for any $y \in H \backslash V_\varepsilon$ there exists $x \in V$ such that $x \succ y$.

It is clear that if a closed A is a ε-solution in the game $G = (I, v, H)$ for any $\varepsilon > 0$, the A is a solution.

Consider the function

$$L(x, y) = \max_{S: x, y \in v(S)} \min_{i \in S} (x_i - y_i).$$

It is clear that $L(x, y) > 0 \Leftrightarrow x \succ y$. Let $\mathcal{A}(G) = \{B \in 2^H : B \,, \max_{x, y \in B} L(x, y) = 0\}$.

Proposition 4.1. *Let $\varepsilon > 0$. The game $G = (I, v, H)$ has a ε-solution if and only if*

$$\sup_{A \in \mathcal{A}(G)} \{ \min_{y \in H \backslash A_\varepsilon} \max_{x \in A} L(x, y) \} > 0. \tag{17}$$

Proof. Let (17) holds. Then there exist $A \subseteq H$ such that

$$\min_{y \in H \backslash A_\varepsilon} \max_{x \in A} L(x, y) > 0 \tag{18}$$

and

$$\max_{x, y \in A} L(x, y) = 0. \tag{19}$$

It follows from (18) that $\max_{x \in A} L(x, \overline{y}) > 0$ for each $\overline{y} \in H \backslash A_\varepsilon$. Consequently for arbitrary $\overline{y} \in H \backslash A_\varepsilon$ one can find a $\overline{x} \in A$ such that $L(\overline{x}, \overline{y}) > 0$, i.e. $\overline{x} \succ \overline{y}$. It now follows from (19) that $L(x, y) \leq 0$ for all $x, y \in A$, i.e. neither x dominates y, nor vice versa. Consequently the set A is a ε-solution.

Now let A is a ε-solution. Then $L(x, y) \leq 0$ for all $x, y \in A$, but $L(x, x) = 0$ implying (19).

With A being a ε-solution, for any $\overline{y} \in H \backslash A_\varepsilon$ there exists a $\overline{x} \in A$ such that $\overline{x} \succ \overline{y}$, i.e. $L(\overline{x}, \overline{y}) > 0$. Consequently $\max_{x \in A} L(x, \overline{y}) > 0$ for any $\overline{y} \in H \backslash A_\varepsilon$ and therefore

$$\min_{y \in H \backslash A_\varepsilon} \max_{x \in A} L(x, y) > 0.$$

$\square$

Let us choose now a $\varepsilon > 0$ and introduce a two-person zero-sum game $\Gamma_\varepsilon(I, v, H)$, in which the first player makes a first move by choosing a $A \in \mathcal{A}$ and then the second player replies by choosing $y \in H \backslash A_\varepsilon$. Finally the first player makes the third (and the last) move by choosing $x \in A$. The income of the first player in this game equals $L(x, y)$ (the second one gets $-L(x, y)$).

The following statement is a direct corollary of Proposition 4.1.

Proposition 4.2. *The game $G = (I, v, H)$ has a ε-solution if and only if the maximal guaranteed gain of the first plater in $\Gamma_\varepsilon(I, v, H)$ is positive. Moreover any $A \in \mathcal{A}$ yielding such gain is a ε-solution to G.*

One can now define a two-player zero-sum game in the normal form $N_\varepsilon(I, v, H)$, where the strategies of the first players are the sets $A \in \mathcal{A}$ and the strategies of the second player are the mappings $f : A \in \mathcal{A} \mapsto H \backslash A_\varepsilon$. Let F denote the set of the strategies of the second player. Let the payoff to the first player in this game $N_\varepsilon(I, v, H)$ be

$$h(A, f) = \max_{x \in A} L(x, f(A)).$$

Proposition 4.3. *The game $N_\varepsilon(I, v, H)$ has a value, i.e.*

$$\sup_{A \in \mathcal{A}} \inf_{f \in F} h(A, f) = \inf_{f \in F} \sup_{A \in \mathcal{A}} h(A, f).$$

Proof. Consider the strategy f^* such that

$$\min_{y \in H \backslash A_\varepsilon} \max_{x \in A} L(x, y) = \max_{x \in A} L(x, f^*(A))$$

for any $A \in \mathcal{A}$.

Then

$$\sup_{A \in \mathcal{A}} \inf_{f \in F} h(A, f) \leq \inf_{f \in F} \sup_{A \in \mathcal{A}} h(A, f) \leq \sup_{A \in \mathcal{A}} h(A, f^*) =$$

$$= \sup_{A \in \mathcal{A}} \max_{x \in A} L(x, f^*(A)) = \sup_{A \in \mathcal{A}} \min_{y \in H \backslash A_\varepsilon} \max_{x \in A} L(x, y) = \sup_{A \in \mathcal{A}} \inf_{f \in F} h(A, f).$$

$\square$

Theorem 4.1. *([11]) The game $G = (I, v, H)$ has a ε-solution if and only if*

$$\sup_{A \in \mathcal{A}} \inf_{f \in F} h(A, f) > 0$$

holds for the game $N_\varepsilon(I, v, H)$.

Proof. Follows from Proposition 4.1, 4.3.

$\square$

5 Stability for two-action multi-agent games

One of the most popular game in recent game-theoretic literature is the so called minority game, see [4]. It represents a particular game of many agents with each agent having two strategies. With this motivation, we are going to analyze here the stability property of games with two actions of each player.

To prepare the stage for our analysis, let us consider a general non-symmetric game Γ of m agents, where each agent j has n_j strategies $\{s_j^k\}$, $k = 1, \cdots, n_j$, and receives payoffs $\Pi_j(s_1^{j_1}, s_2^{j_2}, \cdots, s_m^{j_m})$ at a profile $(s_1^{j_1}, s_2^{j_2}, \cdots, s_m^{j_m})$. Then, for a mixed strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m)$ with

$$\sigma_1 = (x_1^1, ..., x_{n_1}^1), \quad \sigma_2 = (x_1^2, ..., x_{n_2}^2), \quad ..., \quad \sigma_m = (x_1^m, ..., x_{n_m}^m)$$

the payoff of agent j becomes

$$\Pi_i(\sigma_1, \sigma_2, ..., \sigma_m) = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} ... \sum_{j_m=1}^{n_m} x_{j_1}^1 x_{j_2}^2 ... x_{j_m}^m \Pi_i(s_1^{j_1}, s_2^{j_2}, ..., s_m^{j_m}). \tag{20}$$

The Replicator Dynamics (RD) equations have the form

$$\dot{x}_i^j = (\Pi_j(s_j^i, \sigma_{-j}) - \Pi_j(\sigma_j, \sigma_{-j}))x_i^j, \quad i = 1, ..., n_j, \; j = 1, ..., m, \tag{21}$$

where σ_{-j} denotes the collection of the strategies of all players in σ others than j. This system describes an evolution of the behavior of the players applying a try-and-error method of shifting the strategies in the direction of a better payoff. As is well known, a Nash equilibrium is a fixed point of system (21). The r.h.s of (21) is sometimes called the Nash vector field of the game.

Recall that a Nash equilibrium for a game Γ is called *asymptotically stable, neutrally stable or unstable in the Lyapunov sense (or dynamically)* if it so for the corresponding dynamics (21). (This means roughly speaking that if starting with strategies near the equilibrium, players would adjust their strategies in the direction of better payoffs, their strategies would converge to this equilibrium.) On the other hand, a Nash equilibrium $(\sigma_1, ..., \sigma_m)$ in such game is called *structurally stable* if for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that for all games $\tilde{\Gamma}$ with the same number of players and pure strategies and with payoffs $\tilde{\Pi}$ that differ from Π no more than by δ, i.e. such that

$$|\Pi_i(s_1^{j_1}, s_2^{j_2}, ..., s_m^{j_m}) - \tilde{\Pi}_i(s_1^{j_1}, s_2^{j_2}, ..., s_m^{j_m})| < \delta \tag{22}$$

for all $i, s_1^{j_1}, s_2^{j_2}, ..., s_m^{j_m}$, there exists a Nash equilibrium $(\tilde{\sigma}_1, ..., \tilde{\sigma}_m)$ for the game $\tilde{\Gamma}$ such that $|\tilde{\sigma}_j - \sigma_j| < \epsilon$ for all $j = 1, ..., m$. It makes sense also to speak about structural stability of dynamically stable or unstable equilibria, i.e. a dynamically stable (or unstable) Nash equilibrium $(\sigma_1, ..., \sigma_m)$ in a game Γ is called *structurally stable* if for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that for all games $\tilde{\Gamma}$ with the same number of players and pure strategies and with payoffs $\tilde{\Pi}$ that differ from Π no more than by δ (i.e. (22) holds), there exists a dynamically stable (respectively unstable) Nash equilibrium $(\tilde{\sigma}_1, ..., \tilde{\sigma}_m)$ for the game $\tilde{\Gamma}$ such that $|\tilde{\sigma}_j - \sigma_j| < \epsilon$ for all $j = 1, ..., m$.

The notion of stability is closely related to another important notion of a *generic property*: a property (object or characteristics) in a class of structures parametrized by a collection of real numbers s from a given subset S of a Euclidean space is called *generic* if it holds for s from a subset $\tilde{S} \subset S$ that is both open (which means that if $s_0 \in \tilde{S}$, then all $s \in S$ that are closed enough to s_0 belong to $\tilde{S}$ as well, i.e. the property of being in $\tilde{S}$ is structurally stable in any point $s \in \tilde{S}$) and dense (which means that for any s_0 there exists an $s \in \tilde{S}$ that is arbitrary close to s_0, i.e. the negation of being in $\tilde{S}$ is nowhere structurally stable).

As mentioned above, we are going to concentrate on the class Γ_n^2 of mixed strategy extensions of games of n players each having only two strategies.

Let $A^i_{j_1,...,j_n}$ denote the payoff to i under pure profile $\{j_1, ..., j_n\}$, $j_k = 1, 2$. A mixed strategy profile can be described by families

$$\sigma_1 = (x_1, 1 - x_1), \sigma_2 = (x_2, 1 - x_2), ..., \sigma_n = (x_n, 1 - x_n).$$

Equations (21) can be written in terms of $x_1, ..., x_n$ yielding (check it!)

$$\dot{x}_i = x_i(1 - x_i) \sum_{I \in \{1,...,n\}\backslash i} \tilde{A}_I^i \prod_{k \in I} x_k \prod_{k \notin I}(1 - x_k), \quad i = 1, ..., n, \tag{23}$$

where

$$\tilde{A}_I^i = A^i_{j_1...j_{i-1}1j_{i+1}...j_n} - A^i_{j_1...j_{i-1}2j_{i+1}...j_n}$$

with $j_k = 1$ whenever $k \in I$ and $j_k = 2$ otherwise. Hence pure mixed (i.e. with all probabilities being positive) Nash equilibria for a game in Γ_n^2 are given by vectors $x^\star = (x_1^\star, ..., x_n^\star)$ with coordinates from $(0, 1)$ solving the following system of n equations

$$\sum_{I \in \{1,...,n\}\backslash i} \tilde{A}_I^i \prod_{k \in I} x_k \prod_{k \notin I}(1 - x_k) = 0, \quad i = 1, ..., n. \tag{24}$$

In particular for $n = 3$, denoting x_1, x_2, x_3 by x, y, z and arrows of payoffs A^1, A^2, A^3 by A, B, C yields for system (23) the following explicit form

$$\begin{cases} \dot{x} = x(1 - x)(a + A_2 y + A_3 z + Ayz), & 0 \leq x \leq 1 \\ \dot{y} = y(1 - y)(b + B_1 x + B_3 z + Bxz), & 0 \leq y \leq 1 \\ \dot{z} = z(1 - z)(c + C_1 x + C_2 y + Cxy), & 0 \leq z \leq 1 \end{cases} \tag{25}$$

as well as the form

$$\begin{cases} a + A_2 y + A_3 z + Ayz = 0 \\ b + B_1 x + B_3 z + Bxz = 0 \\ c + C_1 x + C_2 y + Cxy = 0 \end{cases} \tag{26}$$

for system (24), where

$$a = A_{122} - A_{222}, \quad A_2 = A_{112} - A_{212} - a, \; A_3 = A_{121} - A_{221} - a,$$

$$A = A_{111} - A_{211} - a - A_2 - A_3,$$

and the coefficients in other two lines are defined analogously.

Assuming $x^\star = (x_1^\star, ..., x_n^\star)$ solves (24), it is convenient to rewrite system (23) in terms of the deviations from the equilibrium $\xi_i = x_i - x_i^\star$. One then sees by inspection that the matrix of linear approximation (the Jacobian matrix) $J^\star$ has the entries

$$J_{ij}^\star = x_i^\star(1 - x_i^\star) \sum_{I \in \{1,...,n\}\backslash\{i,j\}} (\tilde{A}_{I\cup j}^i - \tilde{A}_I^i) \prod_{k \in I} x_k^\star \prod_{k \notin (I \cup \{i,j\})}(1 - x_k^\star) \tag{27}$$

for $i \neq j$ and $J_{ii}^\star = 0$ for all i.

Theorem 5.1. *Let $J^\star$ be the Jacobian matrix (described in Exercise 9.12) of a pure mixed equilibrium $x^\star = (x_1^\star, ..., x_n^\star)$ solving (24). If at least one of the eigenvalues of $J^\star$ has a non-vanishing real part, then $x^\star$ is unstable in Lyapunov sense (i.e. dynamically). In particular, if n is an odd number, then a necessary condition for the Liapunov stability of $x^\star$ is the degeneracy of $J^\star$, that is $\det J^\star = 0$.*

Proof. From (27) one deduces that $J^\star$ has zeros on the main diagonal. Hence the sum of its eigenvalues vanishes. Consequently if there exists an eigenvalue with a non-vanishing real part there should necessarily exist also an eigenvalue with a positive real part, which implies instability. As eigenvalues with vanishing real parts appear as pairs of conjugate imaginary numbers, it follows that in case of odd n the fact that all real parts vanish implies that zero should be an eigenvalue, i.e. the degeneracy of $J^\star$. □

Let us write down the condition $\det J^\star = 0$ explicitly (i.e. in terms of the payoff coefficients) for the case $n = 3$.

(i) From (27) the condition $\det J^\star = 0$ writes down as

$$(A_2 + Az^\star)(B_3 + Bx^\star)(C_1 + Cy^\star) + (B_1 + Bz^\star)(C_2 + Cx^\star)(A_3 + Ay^\star) = 0. \qquad (28)$$

(ii) Solving (26) by expressing y, z, in terms of x and putting this in the first equation leads to the quadratic equation:

$$v(x^\star)^2 + ux^\star + w = 0, \qquad (29)$$

where

$$w = aC_2B_3 + cbA - bA_3C_2 - cA_2B_3, v = aBC + AB_1C_1 - BA_2C_1 - CA_3B_1,$$

$$u = a(BC_2 + CB_3) + b(AC_1 - CA_3) + c(AB_1 - BA_2) - A_2B_3C_1 - A_3B_1C_2.$$

(iii) Using the system that is solved by $x^\star, y^\star, z^\star$ to express $y^\star z^\star$ as a linear function of $y^\star, z^\star$ allows to rewrite (28) as a quadratic equation (in $x^\star, y^\star, z^\star$). Expressing the quadratic terms of this equation again via linear terms leads to the equation on $x^\star$ only:

$$u + 2x^\star v = 0.$$

Comparing this with (29) leads to the conclusion that (28) is equivalent to the equation

$$u^2 - 4vw = 0, \qquad (30)$$

which is a polynomial homogeneous equation of the sixth order in coefficients a, A, A_2, A_3, b, B, B_1, B_3, c, C, C_1, C_2.

Corollary. The property to have a dynamically unstable pure mixed equilibrium is generic among the games of type Γ_3^2 that have pure mixed equilibria. More precisely, apart from the games from the (algebraic of the sixth order) manifold M described by equation (30) pure mixed equilibria are always dynamically unstable and structurally stable (as dynamically unstable equilibria).

Proposition 5.1. *Under the assumption* (30) *(or equivalently* (28)*) there exist* α, β, γ *such that a function V of the (relative entropy) form*

$$\alpha[x^\star \ln x + (1-x^\star)\ln(1-x)] + \beta[y^\star \ln y + (1-y^\star)\ln(1-y)] + \gamma[z^\star \ln z + (1-z^\star)\ln(1-z)] \quad (31)$$

is a first integral for (25)*, i.e. it does not change along the trajectories of* (25) *($dV/dt = 0$) if and only if the condition*

$$AB_1C_1 + aBC = BA_2C_1 + CA_3B_1 \qquad (32)$$

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holds. In the latter case

$$V = (B_1 + Bz^\star)(C_1 + Cy^\star)[x^\star \ln x + (1 - x^\star) \ln(1 - x)]$$

$$-(A_2 + Az^\star)(C_1 + Cy^\star)[y^\star \ln y + (1 - y^\star) \ln(1 - y)]$$

$$-(A_3 + Ay^\star)(B_1 + Bz^\star)[z^\star \ln z + (1 - z^\star) \ln(1 - z)]$$

is an integral.

Proof. Substituting (31) in the equation $dV/dt = 0$ that (31) is an integral of motion (25) if and only if

$$\begin{cases} \alpha(A_2 + Az) + \beta(B_1 + Bz) = 0 \\ \alpha(A_3 + Ay) + \gamma(C_1 + Cy) = 0 \\ \beta(B_3 + Bx) + \gamma(C_2 + Cy) = 0 \\ \alpha A + \beta B + \gamma C = 0. \end{cases}$$

Expressing β, γ in terms of α from the first two equations one observes that the third equation is then automatically satisfied due to (28). Solving the last equation leads to (32). $\qquad\square$

Having a first integral V as above, allows one to conclude that the equilibrium $(x^\star, y^\star, z^\star)$ is neutrally stable in the Lyapunov sense.

6 Nonlinear Markov games

Nonlinear Markov games arise as a (competitive) controlled version of nonlinear Markov processes (an emerging field of intensive research, see e.g. [8], [6] and references therein). This class of games can model a variety of situation for economics and epidemics, statistical physics, and pursuit - evasion processes.

A discrete-time, discrete-space *nonlinear Markov semigroup* Φ^k, $k \in \mathbf{N}$, is specified by an arbitrary continuous mapping $\Phi : \Sigma_n \to \Sigma_n$, where the simplex

$$\Sigma_n = \{\mu = (\mu_1, ..., \mu_n) \in \mathbf{R}_+^n : \sum_{i=1}^n \mu_i = 1\}$$

represents the set of probability laws on the finite state space $\{1, ..., n\}$. For a measure $\mu \in \Sigma_n$ the family $\mu^k = \Phi^k \mu$ can be considered an evolution of measures on $\{1, ..., n\}$. But it does not yet define a random process, because finite-dimensional distributions are not specified. In order to obtain a process we have to choose a *stochastic representation* for Φ, i.e. to write it down in the form

$$\Phi(\mu) = \{\Phi_j(\mu)\}_{j=1}^n = \{\sum_{i=1}^n P_{ij}(\mu)\mu_i\}_{j=1}^n, \tag{33}$$

where $P_{ij}(\mu)$ is a family of stochastic matrices depending on μ (nonlinearity!), whose elements specify the *nonlinear transition probabilities*. For any given $\Phi : \Sigma_n \mapsto \Sigma_n$ a representation (33) exists but is not unique. There exists a unique representation (33) with the additional condition that all matrices $P_{ij}(\mu)$ are one dimensional:

$$P_{ij}(\mu) = \Phi_j(\mu), \quad i, j = 1, ..., n. \tag{34}$$

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Once a stochastic representation (33) for a mapping Φ is chosen we can naturally define, for any initial probability law $\mu = \mu^0$, a stochastic process i_l, $l \in \mathbf{Z}_+$, called a *nonlinear Markov chain*, on $\{1, ..., n\}$ in the following way. Starting with an initial position i_0 distributed according to μ we then choose the next point i_1 according to the law $\{P_{i_0 j}(\mu)\}_{j=1}^n$, the distribution of i_1 becoming $\mu^1 = \Phi(\mu)$:

$$\mu_j^1 = \mathbf{P}(i_1 = j) = \sum_{i=1}^n P_{ij}(\mu)\mu_i = \Phi_j(\mu).$$

Then we choose i_2 according to the law $\{P_{i_1 j}(\mu^1)\}_{j=1}^n$, and so on. The law of this process at any given time k is $\mu^k = \Phi^k(\mu)$; that is, it is given by the semigroup. However, now the finite-dimensional distributions are defined as well. Namely, say for a function f of two discrete variables, we have

$$\mathbf{E}f(i_k, i_{k+1}) = \sum_{i,j=1}^n f(i,j)\mu_i^k P_{ij}(\mu^k).$$

In other words, this process can be defined as a time nonhomogeneous Markov chain with transition probabilities $P_{ij}(\mu^k)$ at time $t = k$.

We turn now to nonlinear chains in continuous time. A *nonlinear Markov semigroup* in continuous time and with finite state space $\{1, ..., n\}$ is defined as a semigroup Φ^t, $t \geq 0$, of continuous transformations of Σ_n. As in the case of discrete time the semigroup itself does not specify a process. To get a process, assume the semigroup Φ^t is differentiable in t, so that we can define the *(nonlinear) infinitesimal generator* of the semigroup Φ^t as the nonlinear operator on measures given by

$$A(\mu) = \frac{d}{dt}\Phi^t|_{t=0}(\mu).$$

The semigroup identity for Φ^t implies that $\Phi^t(\mu)$ solves the Cauchy problem

$$\frac{d}{dt}\Phi^t(\mu) = A(\Phi^t(\mu)), \quad \Phi^0(\mu) = \mu. \tag{35}$$

As follows from the invariance of Σ_n under these dynamics, the mapping A is *conditionally positive* in the sense that $\mu_i = 0$ for a $\mu \in \Sigma_n$ implies $A_i(\mu) \geq 0$ and is also *conservative* in the sense that A maps the measures from Σ_n to the space of signed measures

$$\Sigma_n^0 = \{\nu \in \mathbf{R}^n : \sum_{i=1}^n \nu_i = 0\}.$$

We shall say that such a generator A has a *stochastic representation* if it can be written in the form

$$A_j(\mu) = \sum_{i=1}^n \mu_i Q_{ij}(\mu) = (\mu Q(\mu))_j, \tag{36}$$

where $Q(\mu) = \{Q_{ij}(\mu)\}$ is a family of infinitesimally stochastic matrices (or Q=matrices) depending on $\mu \in \Sigma_n$. Thus in its stochastic representation the generator has the form of a usual Markov chain generator, though depending additionally on the present distribution.

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The existence of a stochastic representation for the generator is not difficult to obtain, see [8].

The examples of nonlinear Markov chains are numerous including Lotka-Volterra systems, general replicator dynamics of the evolutionary game theory, models of epidemics, coagulation processes, see more in [8].

Now we discuss the corresponding nonlinear extension of controlled processes.

Nonlinear Markov games can be considered as a systematic tool for modeling deception. In particular, in a game of pursuit - evasion, an evading object can create false objectives or hide in order to deceive the pursuit. Thus, observing this object leads not to its precise location, but to its distribution only, implying that it is necessary to build competitive control on the basis of the distribution of the present state. Moreover, by observing the action of the evading objects, one can make conclusions about its certain dynamic characteristics making the (predicted) transition probabilities depending on the observed distribution via these characteristics. This is precisely the type of situations modeled by nonlinear Markov games.

The starting point for the analysis is the observation that a nonlinear Markov semigroup is after all just a deterministic dynamic system (though on a weird state space of measures and with a specifically structured payoff function). Thus, as the stochastic control theory is a natural extension of the deterministic control, we are going to further extend it by turning back to deterministic control, but of measures, thus exemplifying the usual spiral development of science. The next 'turn of the screw' would lead to stochastic measure-valued games forming a stochastic control counterpart for the class of processes discussed in the previous section.

We shall work directly in the competitive control setting (game theory), which of course includes the usual optimization as a particular case, but for simplicity only in discrete time and finite original state space $\{1, ..., n\}$. The full state space is then chosen as a set of probability measures Σ_n on $\{1, ..., n\}$.

Suppose we are given two metric spaces U, V of the control parameters of two players, a continuous transition cost function $g(u, v, \mu)$, $u \in U$, $v \in V$, $\mu \in \Sigma_n$ and a transition law $\nu(u, v, \mu)$ prescribing the new state $\nu \in \Sigma_n$ obtained from μ once the players had chosen their strategies $u \in U, v \in V$. The problem of the corresponding one-step game (with sequential moves) consists in calculating the Bellman operator

$$(BS)(\mu) = \min_u \max_v [g(u, v, \mu) + S(\nu(u, v, \mu))] \tag{37}$$

for a given final cost function S on Σ_n. According to the dynamic programming principle (see e.g. [10]), the dynamic multi-step game solution is given by the iterations $B^k S$. Often of interest is the behavior of this optimal cost $B^k S(\mu)$ as the number of steps k go to infinity.

The function $\nu(u, v, \mu)$ can be interpreted as the controlled version of the mapping Φ specifying a nonlinear discrete time Markov semigroup. Assume a stochastic representation for this mapping is chosen, i.e.

$$\nu_j(u, v, \mu) = \sum_{i=1}^{n} \mu_i P_{ij}(u, v, \mu)$$

with a given family of (controlled) stochastic matrices P_{ij}. Then it is natural to assume

g to describe the average over the random transitions, i.e. be given by

$$g(u, v, \mu) = \sum_{i,j=1}^{n} \mu_i P_{ij}(u, v, \mu) g_{ij}$$

with certain real coefficients g_{ij}. Under this assumption the Bellman operator (37) takes the form

$$(BS)(\mu) = \min_u \max_v [\sum_{i,j=1}^{n} \mu_i P_{ij}(u, v, \mu) g_{ij} + S\left(\sum_{i=1}^{n} \mu_i P_{i.}(u, v, \mu)\right)]. \tag{38}$$

We can now identify the (not so obvious) place of the usual stochastic control theory in this nonlinear setting. Namely, assume P_{ij} above do not depend on μ. But even then the set of the linear functions $S(\mu) = \sum_{i=1}^{n} s_i \mu^i$ on measures (identified with the set of vectors $S = (s_1, ..., s_n)$) is not invariant under B. Hence we are not automatically reduced to the usual stochastic control setting, but to a game with incomplete information, where the states are probability laws on $\{1, ..., n\}$, i.e. when choosing a move the players do not know the position precisely, but only its distribution. Only if we allow only Dirac measures μ as a state space (i.e. no uncertainty on the state), the Bellman operator would be reduced to the usual one of the stochastic game theory:

$$(\bar{B}S)_i = \min_u \max_v \sum_{j=1}^{n} P_{ij}(u, v)(g_{ij} + S_j). \tag{39}$$

As an example of a nonlinear result we shall get here an analog of the result on the existence of the average income for long lasting games (see [8] for a proof).

Proposition 6.1. *If the mapping ν is a contraction uniformly in u, v, i.e. if*

$$\|\nu(u, v, \mu^1) - \nu(u, v, \mu^2)\| \le \delta \|\mu^1 - \mu^2\| \tag{40}$$

with a $\delta \in (0, 1)$, where $\|\nu\| = \sum_{i=1}^{n} |\nu_i|$, and if g is Lipschitz continuous, i.e.

$$\|g(u, v, \mu^1) - g(u, v, \mu^2)\| \le C \|\mu^1 - \mu^2\| \tag{41}$$

with a constant $C > 0$, then there exists a unique $\lambda \in \mathbf{R}$ and a Lipschitz continuous function S on Σ_n such that

$$B(S) = \lambda + S, \tag{42}$$

and for all $g \in C(\Sigma_n)$ we have

$$\|B^m g - m\lambda\| \le \|S\| + \|S - g\|, \tag{43}$$

$$\lim_{m \to \infty} \frac{B^m g}{m} = \lambda. \tag{44}$$

One can extend the other results for stochastic multi-step games to this nonlinear setting, say, the turnpike theorems from [10], and then go on studying the nonlinear Markov analogs of differential games.

7 Colored options as a game against Nature

Here we introduce a game-theoretic analysis of rainbow options in incomplete markets. Further developments can be found in [12], [3] and [9].

Recall that a European option is a contract between two parties where one party has right to complete a transaction in the future (with previously agreed amount, date and price) if he/ she chooses, but is not obliged to do so. More precisely, consider a financial market dealing with several securities: the risk-free bonds (or bank account) and J common stocks, $J = 1, 2...$ In case $J > 1$, the corresponding options are called *colored or rainbow options* (J-colors option for a given J). Suppose the prices of the units of these securities, B_m and S_m^i, $i \in \{1, 2, ..., J\}$, change in discrete moments of time $m = 1, 2, ...$ according to the recurrent equations $B_{m+1} = \rho B_m$, where the $\rho \geq 1$ is an interest rate which remains unchanged over time, and $S_{m+1}^i = \xi_{m+1}^i S_m^i$, where $\xi_m^i, i \in \{1, 2, ..., J\}$, are unknown sequences taking values in some fixed intervals $M_i = [d_i, u_i] \subset \mathbf{R}$. This model generalizes the colored version of the classical CRR model in a natural way. In the latter a sequence ξ_m^i is confined to take values only among two boundary points d_i, u_i, and it is supposed to be random with some given distribution. In our model any value in the interval $[d_i, u_i]$ is allowed and no probabilistic assumptions are made.

The type of an option is specified by a given premium function f of J variables. The following are the standard examples:

option delivering the best of J risky assets and cash

$$f(S^1, S^2, ..., S^J) = \max(S^1, S^2, ..., S^J, K), \tag{45}$$

calls on the maximum of J risky assets

$$f(S^1, S^2, ..., S^J) = \max(0, \max(S^1, S^2, ..., S^J) - K), \tag{46}$$

multiple-strike options

$$f(S^1, S^2, ..., S^J) = \max(0, S^1 - K_1, S^2 - K_2, ..., S^J - K_J), \tag{47}$$

portfolio options

$$f(S^1, S^2, ..., S^J) = \max(0, n_1 S^1 + n_2 S^2 + ... + n_J S^J - K), \tag{48}$$

and spread options

$$f(S^1, S^2) = \max(0, (S^2 - S^1) - K). \tag{49}$$

Here, the $S^1, S^2, ..., S^J$ represent the (in principle unknown at the start) expiration date values of the underlying assets, and $K, K_1, ..., K_J$ represent the (agreed from the beginning) strike prices. The presence of max in all these formulae reflects the basic assumption that the buyer is not obliged to exercise his/her right and would do it only in case of a positive gain.

The investor is supposed to control the growth of his/her capital in the following way. Let X_m denote the capital of the investor at the time $m = 1, 2, ...$ At each time $m-1$ the investor determines his portfolio by choosing the numbers γ_m^i of common stocks of each kind to be held so that the structure of the capital is represented by the formula

$$X_{m-1} = \sum_{j=1}^{J} \gamma_m^j S_{m-1}^j + (X_{m-1} - \sum_{j=1}^{J} \gamma_m^j S_{m-1}^j),$$

where the expression in brackets corresponds to the part of his capital held in the bank account. The control parameters γ_m^j can take all real values, i.e. short selling and borrowing are allowed. The value ξ_m becomes known at the moment m and thus the capital at the moment m becomes

$$X_m = \sum_{j=1}^{J} \gamma_m^j \xi_m^j S_{m-1}^j + \rho(X_{m-1} - \sum_{j=1}^{J} \gamma_m^j S_{m-1}^j), \tag{50}$$

if transaction costs are not taken into account.

If n is the prescribed *maturity date*, then this procedures repeats n times starting from some initial capital $X = X_0$ (selling price of an option) and at the end the investor is obliged to pay the premium f to the buyer. Thus the (final) income of the investor equals

$$G(X_n, S_n^1, S_n^2, ..., S_n^J) = X_n - f(S_n^1, S_n^2, ..., S_n^J). \tag{51}$$

The evolution of the capital can thus be described by the n-step game of the investor with Nature, the behavior of the latter being characterized by unknown parameters ξ_m^j. The strategy of the investor is by definition any sequences of vectors $(\gamma_1, \cdots, \gamma_n)$ such that each γ_m could be chosen using the whole previous information: the sequences $X_0, ..., X_{m-1}$ and $S_0^i, ..., S_{m-1}^j$ (for every stock $j = 1, 2, ..., J$). The control parameters γ_m^j can take all real values, i.e. short selling and borrowing are allowed. A position of the game at any time m is characterized by $J + 1$ non-negative numbers $X_m, S_m^1, \cdots, S_m^J$ with the final income specified by the function

$$G(X, S^1, ..., S^J) = X - f(S^1, ..., S^J). \tag{52}$$

The main definition of the theory is as follows. A strategy $\gamma_1, \cdots, \gamma_n$, of the investor is called a *hedge*, if for any sequence $(\xi_1, \cdots, \xi_n)$ the investor is able to meet his/her obligations, i.e.
$$G(X_n, S_n^1, ..., S_n^J) \geq 0.$$
The minimal value of the capital X_0 for which the hedge exists is called the *hedging price* H of an option.

Looking for the guaranteed payoffs means looking for the worst-case scenario (so called *robust-control approach*), i.e. for the minimax strategies. Thus if the final income is specified by a function G, the guaranteed income of the investor in a one-step game with the initial conditions $X, S^1, ..., S^J$ is given by the *Bellman operator*

$$\mathbf{B}G(X, S^1, \cdots, S^J) = \frac{1}{\rho} \max_{\gamma} \min_{\{\xi^j \in [d_j, u_j]\}} G(\rho X + \sum_{i=1}^{J} \gamma^i \xi^i S^i - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, \cdots, \xi^J S^J), \tag{53}$$

and (as it follows from the standard backward induction argument) the guaranteed income of the investor in the n-step game with the initial conditions $X_0, S_0^1, ..., S_0^J$ is given by the formula

$$\mathbf{B}^n G(X_0, S_0^1, ..., S_0^J).$$

In our model G is given by (52). Clearly for G of the form

$$G(X, S^1, \cdots, S^J) = X - f(S^1, \cdots, S^J),$$

$$\mathbf{B}G(X, S^1, ..., S^J) = X - \frac{1}{\rho} \min_\gamma \max_\xi [f(\xi^1 S^1, \xi^2 S^2, \cdots, \xi^J S^J) - \sum_{j=1}^J \gamma^j S^j (\xi^j - \rho)],$$

and hence

$$\mathbf{B}^n G(X, S^1, \cdots, S^J) = X - \frac{1}{\rho^n} (\mathcal{B}^n f)(S^1, \cdots, S^J),$$

where the *reduced Bellman operator* is defined as:

$$(\mathcal{B}f)(z^1, ..., z^J) = \min_\gamma \max_{\{\xi^j \in [d_j, u_j]\}} [f(\xi^1 z^1, \xi^2 z^2, \cdots, \xi^J z^J) - \sum_{j=1}^J \gamma^j z^j (\xi^j - \rho)], \quad (54)$$

or, more concisely,

$$(\mathcal{B}f)(z) = \min_\gamma \max_{\{\xi^j \in [d_j, u_j]\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)]. \quad (55)$$

This leads to the following result from [7].

Theorem 7.1. *The minimal value of X_0 for which the income of the investor is non-negative (and which by definition is the hedge price H^n in the n-step game) is given by*

$$H^n = \frac{1}{\rho^n} (\mathcal{B}^n f)(S_0^1, \cdots, S_0^J). \quad (56)$$

We shall now develop a method for evaluating the operator (54) showing in particular how naturally the risk-neutral probability laws appear, as if by miracle, in this evaluation. The proof of all results below (and its various extensions) can be found in [9].

Let us say that a probability law $\mu \in \mathcal{P}(E)$ on a compact set $E \subset \mathbf{R}^d$ is *risk-neutral with respect to the origin*, or shortly, *risk-neutral* if the origin is its barycenter, that is $\int_E \xi \mu(d\xi) = 0$. The set of all risk-neutral laws on E will be denoted by $\mathcal{P}_{rn}(E)$.

Let us say that a finite family of vectors $E = \{\xi_1, \cdots, \xi_k\}$ in $\mathbf{R}^d$ is in *general position* if the vectors of any subset of $\{\xi_1, \cdots, \xi_k\}$ of size d are linearly independent (in particular, all vectors in E are non-vanishing). A subset $E \subset \mathbf{R}^d$ is called (strongly) *positively complete*, if there exists no $\omega \in \mathbf{R}^d$ such that $(\omega, \xi) \geq 0$ for all $\xi \in E$.

Proposition 7.1. *Let a finite set $E = \{\xi_1, \cdots, \xi_{d+1}\}$ be strongly positively complete in $\mathbf{R}^d$. Then the family E is in general position and there exists a unique risk-neutral probability law $\{p_1, \cdots, p_{d+1}\}$ on $\{\xi_1, \cdots, \xi_{d+1}\}$, with*

$$p_i = C^{-1}(-1)^{i-1} \det \begin{pmatrix} \xi_1^1 & \cdots & \xi_{i-1}^1 & \xi_{i+1}^1 & \cdots & \xi_{d+1}^1 \\ \xi_1^2 & \cdots & \xi_{i-1}^2 & \xi_{i+1}^2 & \cdots & \xi_{d+1}^2 \\ & & \cdots & & \\ \xi_1^d & \cdots & \xi_{i-1}^d & \xi_{i+1}^d & \cdots & \xi_{d+1}^d \end{pmatrix}, \quad i = 1, \cdots, d, \quad (57)$$

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where

$$C = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_{d+1}^1 \\ & & \cdots & \\ \xi_1^d & \xi_2^d & \cdots & \xi_{d+1}^d \end{pmatrix}. \tag{58}$$

For arbitrary k we have the following.

Proposition 7.2. *Let a family $E = \{\xi_1, \cdots, \xi_k\}$ be strongly positively complete and in general position. Then the extreme points of the convex set of risk-neutral probabilities on $\{\xi_1, \cdots, \xi_k\}$ are risk-neutral probabilities with supports on strongly positively complete subsets of E of size precisely $d+1$.*

Now we are going to evaluate the minimax expression

$$\Pi[\xi_1, \cdots, \xi_k](f) = \inf_{\gamma \in \mathbf{R}^d} \max_i [f(\xi_i) - (\xi_i, \gamma)]. \tag{59}$$

Theorem 7.2. *Let a family of vectors $\xi_1, \cdots, \xi_k$ in $\mathbf{R}^d$ be strongly positively complete and in general position. Then*

$$\Pi[\xi_1, \cdots, \xi_k](f) = \max_\mu \mathbf{E}_\mu f(\xi), \tag{60}$$

where max is taken over all extreme points μ of risk-neutral laws on $\{\xi_1, \cdots, \xi_k\}$, given by Proposition 7.2, and inf in (59) is attained on a certain finite γ.

Changing variables $\xi = (\xi^1, \ldots, \xi^J)$ to $\eta = \xi \circ z$ yields

$$(\mathcal{B}f)(z^1, ..., z^J) = \frac{1}{\rho} \min_\gamma \max_{\{\eta \in [z^i d_i, z^i u_i]\}} [f(\eta) - \sum_{i=1}^J \gamma^i (\eta^i - \rho z^i)], \tag{61}$$

or, by shifting,

$$(\mathcal{B}f)(z^1, ..., z^J) = \frac{1}{\rho} \min_\gamma \max_{\{\eta \in [z^i(d_i - \rho), z^i(u_i - \rho)]\}} [f(\eta + \rho z) - (\gamma, \eta)]. \tag{62}$$

Assuming f is convex (which is often the case for option payoffs), we can apply Theorem 7.2, where max is taken over the set of vectors

$$\eta_I = \xi_I \circ z - \rho z,$$

being the vertices of the rectangular parallelepiped

$$\Pi_{z, \rho} = \times_{i=1}^J [z^i(d_i - \rho), z^i(u_i - \rho)],$$

where

$$\xi_I = \{d_i|_{i \in I}, u_j|_{j \notin I}\},$$

are the vertices of

$$\Pi = \times_{i=1}^J [d_i, u_i], \tag{63}$$

parametrized by all subsets (including the empty one) $I \subset \{1, \ldots, J\}$.

Since the origin is an internal point of Π (because $d_i < \rho < u_i$), the family $\{\eta_I\}$ is strongly positively complete. The condition of general position is rough in the sense that it is fulfilled for an open dense subset of pairs (d_i, u_i). Applying Theorem 7.2 to (62) and returning back to ξ yields the following.

Theorem 7.3. *If the vertices ξ_I of the parallelepiped Π are in general position in the sense that for any J subsets $I_1, \cdots, I_J$, the vectors $\{\xi_{I_k} - \rho\mathbf{1}\}_{k=1}^J$ are independent in $\mathbf{R}^J$, then*

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \max_{\{\Omega\}} \mathbf{E}_\Omega f(\xi \circ z), \quad z = (z^1, \cdots, z^J), \tag{64}$$

where $\{\Omega\}$ is the collection of all subsets $\Omega = \xi_{I_1}, \cdots, \xi_{I_{J+1}}$ of the set of vertices of Π, of size $J + 1$, such that their convex hull contains $\rho\mathbf{1}$ as an interior point ($\mathbf{1}$ is the vector with all coordinates 1), and where $\mathbf{E}_\Omega$ denotes the expectation with respect to the unique probability law $\{p_I\}$, $\xi_I \in \Omega$, on the set of vertices of Π, which is supported on Ω and is risk-neutral with respect to $\rho\mathbf{1}$, that is

$$\sum_{I \subset \{1,\dots,J\}} p_I \xi_I = \rho\mathbf{1}. \tag{65}$$

Risk-neutrality now corresponds to its usual meaning in finance, i.e. (65) means that all discounted stock prices are martingales.

Notice that the max in (64) is over a finite number of explicit expressions, which is of course a great achievement as compared with initial minmax over an infinite set. In particular, it reduces the calculation of the iterations $\mathcal{B}^n f$ to the calculation on a controlled Markov chain. Let us also stress that the number of eligible Ω in (64) is the number of different pyramids (convex polyhedrons with $J+1$ vertices) with vertices taken from the vertices of Π and containing $\rho\mathbf{1}$ as an interior point. Hence this number can be effectively calculated.

Let us point our some properties of the operator $\mathcal{B}$ given by (64) that are obvious, but important for practical calculations: $\rho\mathcal{B}$ is *non-expansive*:

$$\|\mathcal{B}(f_1) - \mathcal{B}(f_2)\| \le \frac{1}{\rho} \|f_1 - f_2\|,$$

and *homogeneous* (both with respect to addition and multiplication):

$$\rho\mathcal{B}(\lambda + f) = \lambda + \rho\mathcal{B}(f), \quad \mathcal{B}(\lambda f) = \lambda\mathcal{B}(f)$$

for any function f and $\lambda \in \mathbf{R}$ (resp. $\lambda > 0$) for the first (resp second) equation.

Next, if f_p is a power function, that is

$$f_p(z) = (z^1)^{i_1} \cdots (z^J)^{i_J},$$

then $f_p(\xi \circ z) = f_p(\xi) f_p(z)$, implying

$$(\mathcal{B}^n f_p)(z) = ((\mathcal{B}f_p)(\mathbf{1}))^n f_p(z). \tag{66}$$

Therefore, power functions are invariant under $\mathcal{B}$ (up to a multiplication by a constant). Consequently, if for a payoff f one can find a reasonable approximation by a power function, that is there exists a power function f_p such that $\|f - f_p\| \le \epsilon$, then

$$\|\mathcal{B}^n f - \lambda^n f_p\| \le \frac{1}{\rho^n} \|f - f_p\| \le \frac{\epsilon}{\rho^n}, \quad \lambda = (\mathcal{B}f_p)(\mathbf{1}), \tag{67}$$

so that an approximate calculation of $\mathcal{B}^n f$ is reduced to the calculation of one number λ. This implies the following scheme for an approximate evaluation of $\mathcal{B}$: first find the best fit to f in terms of functions $\alpha + f_p$ (where f_p is a power function and α a constant), and then use (67).

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